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ON THE EXISTENCE AND MULTIPLICITY OF TOPOLOGICALLY TWISTING INCOMPRESSIBLE H–HARMONIC MAPS AND A STRUCTURAL H–CONDITION

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Abstract. In this paper we address questions on the existence and multiplicity of solutions to the nonlinear elliptic system in divergence form

$$\begin{cases} \operatorname{div}(H\nabla u) = H_s|\nabla u|^2 u + [\operatorname{cof}\nabla u]\nabla\mathscr{P} & \text{in } \Omega, \\ \det\nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Here $H = H(r,s) > 0$ is a weight function of class $\mathscr{C}^2$ with $H_s = \partial H/\partial s$ and $(r,s) = (|x|,|u|^2)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\mathscr{P} = \mathscr{P}(x)$ is an unknown hydrostatic pressure field and φ is a prescribed boundary map. The system is the Euler-Lagrange equation for a weighted Dirichlet energy subject to a pointwise incompressibility constraint on the admissible maps and arises in diverse fields such as geometric function theory and nonlinear elasticity. Whilst the usual methods of critical point theory drastically fail in this vectorial gradient constrained setting we establish the existence of multiple solutions in certain geometries by way of analysing an associated reduced energy for $\mathbf{SO}(n)$-valued fields, a resulting decoupled PDE system and a structure theorem for irrotational vector fields generated by skew-symmetric matrices. Most notably a crucial "H-*condition*" linking to the system and precisely capturing an extreme dimensional dichotomy in the structure of the solution set is discovered and analysed.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) be a bounded domain with a $\mathscr{C}^1$ boundary $\partial\Omega$ and consider the variational energy integral

$$\mathbb{I}[u;\Omega] := \int_\Omega |\nabla u|^2 \, d\mu(x,u), \tag{1.1}$$

where the Lagrangian is of a *weighted* Dirichlet type. Here $d\mu(x,u) = H(|x|,|u|^2)d\mathscr{L}^n$ for some given fixed $H = H(r,s) > 0$ of class $\mathscr{C}^2$ (called the weight function or weight for short) and the competing maps u are restricted to lie in the space of incompressible Sobolev maps $\mathscr{A}_\varphi(\Omega) := \{u \in \mathscr{W}^{1,2}(\Omega,\mathbb{R}^n) : \det\nabla u = 1 \ a.e. \text{ in } \Omega, \ u \equiv \varphi \text{ on } \partial\Omega\}$. Referring to (1.1), ∇u denotes the gradient of u, an $n \times n$ matrix field in Ω, that here is

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additionally required to satisfy the pointwise incompressibility constraint $\det \nabla u = 1$ in Ω, whilst $|\nabla u|$ denotes the Hilbert-Schmidt norm of ∇u and $\varphi \in \mathscr{C}^1(\partial\Omega;\mathbb{R}^n)$ is a fixed boundary condition. [1] The Euler-Lagrange equation associated with this constrained variational problem is formally seen to be the nonlinear system (*see* [1, 3, 4, 5, 6] for background and related discussion)

$$\mathbf{EL}[u;\mathbb{I},\Omega] := \begin{cases} \mathscr{L}_H[u] = \nabla \mathscr{P} & \text{in } \Omega, \\ \det \nabla u = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathscr{P} = \mathscr{P}(x)$ is an *a priori* unknown hydrostatic pressure corresponding to the incompressibility constraint – the so-called Lagrange multiplier – and the differential operator $\mathscr{L}_H$ takes the explicit and highly nonlinear form

$$\begin{aligned} \mathscr{L}_H[u] =& [\operatorname{cof} \nabla u]^{-1} \{\operatorname{div} \left[H(r,|u|^2)\nabla u\right] - H_s(r,|u|^2)|\nabla u|^2 u\} \\ =& [\nabla u]^t \{H_r(r,|u|^2)\nabla u\, \theta + H_s(r,|u|^2)\nabla u \nabla |u|^2\} \\ &+ H(r,|u|^2)[\nabla u]^t \Delta u - H_s(r,|u|^2)|\nabla u|^2[\nabla u]^t u. \end{aligned} \tag{1.3}$$

Without going into technical details we point out that this system is formally the Euler-Lagrange equation associated with the unconstrained variational energy integral

$$\mathbb{I}_{\mathscr{P}}[u;\Omega] = \int_\Omega \{H(|x|,|u|^2)|\nabla u|^2 - 2\mathscr{P}(x)(\det \nabla u - 1)\}\, dx. \tag{1.4}$$

(Notice that $\mathbb{I}_{\mathscr{P}}[u;\Omega] = \mathbb{I}[u;\Omega]$ whenever $u \in \mathscr{A}_\varphi(\Omega)$.) Furthermore in (1.3), $r = |x|$, $\theta = x|x|^{-1}$ whilst $H_r = \partial H/\partial r$, $H_s = \partial H/\partial s$ are the derivatives of the weight function H with respect to the first and second arguments respectively. By a solution to the system (1.2) we mean a pair $(u,\mathscr{P})$ where u is of class $\mathscr{C}^2(\Omega,\mathbb{R}^n) \cap \mathscr{C}(\overline{\Omega},\mathbb{R}^n)$, $\mathscr{P}$ is of class $\mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$ and the pair satisfy the system (1.2)-(1.3) in the pointwise (classical) sense. If the choice of $\mathscr{P}$ is clear from the context we often abbreviate by saying that u is a solution or synonymously (and by analogy with the weight free unconstrained Dirichlet energy case) that u is an incompressible H*-harmonic* map.

Whilst the methods of critical point theory provide a standard and efficient way of establishing the existence of (multiple) solutions to variational problems, due to the complex nature of the incompressibility constraint on the gradient of the competing maps, here, these methods drastically fail and are not applicable. In more technical terms the space $\mathscr{A}_\varphi(\Omega)$ is far from being a Hilbert or Banach manifold whilst due to the *a priori* unknown regularity of the pressure field $\mathscr{P}$, and integrability of the Jacobian determinant $\det \nabla u$, the unconstrained energy integral $\mathbb{I}_{\mathscr{P}}$ need not be everywhere well-defined, let alone, continuously Frechet differentiable.

Throughout the paper we specialise to the set up where $\Omega \subset \mathbb{R}^n$ is a bounded open annulus, for definiteness, $\Omega = \mathbb{X}_n = \mathbb{X}_n[a,b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ with $b > a > 0$, $\varphi \equiv \mathbf{I}_{\overline{\Omega}}$, *i.e.*, the identity map and $H \in \mathscr{C}^2([a,b]\times\mathbb{R})$ satisfies $H > 0$. The choice of φ is prompted by applications to elasticity and to avoid unnecessary technicalities,

[1] Note that throughout the paper we allow the weight H to depend not only of $r = |x|$ but also on $s = |u|^2$ which allows for more generality and leads to some remarkable consequences.

whilst the choice of domain geometry, is prompted by the well-known uniqueness result for star-shaped domains in [14, 23]. Here we prove existence and multiplicity results for incompressible H-harmonic maps by using a different set of ideas and techniques. As it turns out the structure and multiplicity of such solutions depend heavily on the dimensional parity (n being even or odd) as well as a crucial H-condition relating to the weight function H. Indeed it is the significance of this H-condition and the role of (x,u)-dependence in the Lagrangian leading to it that is the most notable phenomenon and the main highlight of the paper.

To this end let us begin by introducing some notation and terminology. Any self-map $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$ can be decomposed into a radial part $\mathscr{R}_u$ and a spherical part $\mathscr{S}_u$:

$$\mathscr{R}_u := |u| \in \mathscr{C}(\overline{\mathbb{X}}_n, [a,b]), \qquad \mathscr{S}_u := u|u|^{-1} \in \mathscr{C}(\overline{\mathbb{X}}_n, \mathbb{S}^{n-1}). \tag{1.5}$$

If $u \equiv x$ on $\partial\mathbb{X}_n$ then $\mathscr{R}_u \equiv a$ and $\mathscr{R}_u \equiv b$ on the inner and outer components of $\partial\mathbb{X}_n$ respectively whilst $\mathscr{S}_u \equiv \theta$ on $\partial\mathbb{X}_n$. Now due to the topological product structure of $\mathbb{X}_n$, the spherical part $\mathscr{S}_u$ can be seen, with a slight abuse of notation, to verify $\mathscr{S}_u \in \mathscr{C}([a,b]; \mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1)$ with $\mathscr{S}_u|_{r=a} = \mathbf{I}_{\mathbb{S}^{n-1}}$ and $\mathscr{S}_u|_{r=b} = \mathbf{I}_{\mathbb{S}^{n-1}}$ where $\mathbf{I}_{\mathbb{S}^{n-1}}$ denotes the identity map of the unit sphere. [2] As a result $\mathscr{S}_u$ represents an element of the fundamental group (*see* [25, 26])

$$\pi_1[\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1)] \cong \pi_1[\mathbf{SO}(n)] \cong \begin{cases} \mathbb{Z}, & n = 2, \\ \mathbb{Z}_2, & n \geqslant 3. \end{cases} \tag{1.6}$$

Conversely any map $\mathscr{S} = \mathscr{S}(r)$ in $\mathscr{C}([a,b]; \mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 1)$ satisfying $\mathscr{S}(a) = \mathscr{S}(b) = \mathbf{I}_{\mathbb{S}^{n-1}}$ gives rise to a self-map $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$ with $u \equiv x$ on $\partial\mathbb{X}_n$ through the recipe $\mathscr{R}_u(x) = f(|x|)$ and $\mathscr{S}_u \equiv \mathscr{S}$, specifically,

$$u : (r, \theta) \mapsto (f(r), \mathscr{S}(r)[\theta]). \tag{1.7}$$

Here $f \in \mathscr{C}([a,b],[a,b])$ is any function satisfying $f(a) = a$ and $f(b) = b$ (*e.g.*, $f(r) \equiv r$). Moving next to the incompressibility constraint it is seen that subject to the differentiability of the radial and spherical parts of u, [3]

$$\nabla u = \mathscr{R}_u \nabla \mathscr{S}_u + \mathscr{S}_u \otimes \nabla \mathscr{R}_u, \tag{1.8}$$

and so u is incompressible *iff* $\det[\mathscr{R}_u \nabla \mathscr{S}_u + \mathscr{S}_u \otimes \nabla \mathscr{R}_u] = 1$ (*see* below for more). With this notation in place a topologically twisting incompressible H-harmonic map by definition is a twice continuously differentiable self-map u with spherical part $\mathscr{S}_u$ resulting from a suitable $\mathbf{SO}(n)$-valued field $\mathbf{Q}$ as described in (a)-(b) below such that the pair $(u, \mathscr{P})$ for a suitable choice of $\mathscr{P}$ forms a solution to the system (1.2).

[2] Throughout the paper $\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = d)$ $(d \in \mathbb{Z})$ denotes the connected component of the mapping space $\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$ consisting of maps with Hopf degree d. It is a well known fact that these components are of different homotopy types. For instance in contrast to (1.6) for $d = 0$ we have $\pi_1[\mathscr{C}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1}; \mathbf{deg} = 0)] \cong \pi_1(\mathbb{S}^{n-1}) \oplus \pi_n(\mathbb{S}^{n-1})$ with the latter being $\cong \mathbb{Z}$ (for $n = 2$ or $n = 3$), and $\mathbb{Z}_2$ (for $n \geqslant 4$). For more on this *see* [8, 9, 30, 31].

[3] Note that since the spherical part $\mathscr{S}_u$ maps into the sphere we have $\det \nabla \mathscr{S}_u = 0$.

(a) **Twists** $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$**.** By a generalised twist or simply a *twist* we understand a self-map u whose radial and spherical parts are given by

$$\mathscr{R}_u(x) = |x|, \qquad \mathscr{S}_u(x) = \mathbf{Q}(|x|)x|x|^{-1}, \qquad x \in \overline{\mathbb{X}}_n. \tag{1.9}$$

Here the curve $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n))$ is referred to as the twist path associated with u. In order to ensure $u \equiv x$ on $\partial\Omega = \partial\mathbb{X}_n$ we set $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. In this event the twist path is a closed curve in $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ thus representing an element of $\pi_1[\mathbf{SO}(n)] \cong \mathbb{Z}_2$ $(n \geqslant 3)$ and $\cong \mathbb{Z}$ $(n = 2)$. Here we refer to $\mathbf{Q} = \mathbf{Q}(r)$ as the twist *loop* associated to u. It can be seen that subject to the differentiability of the twist path $\mathbf{Q}$, we have

$$\nabla\mathscr{R}_u = \theta, \qquad \nabla\mathscr{S}_u = \frac{1}{r}[\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta], \tag{1.10}$$

(with $\dot{\mathbf{Q}} = d\mathbf{Q}/dr$) and so using (1.8), $\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$ and thus $\det \nabla u = 1$.
(b) **Whirls** $u \in \mathscr{C}(\overline{\mathbb{X}}_n, \overline{\mathbb{X}}_n)$**.** By a whirl map or *whirl* for simplicity we understand a self-map u whose radial and spherical parts have the forms

$$\mathscr{R}_u(x) = |x|, \qquad \mathscr{S}_u(x) = \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}, \qquad x \in \overline{\mathbb{X}}_n. \tag{1.11}$$

Here we denote by $\rho = \rho(x)$ the vector of 2-plane radial variables $(\rho_1, \dots, \rho_N)$, defined, depending on the dimension $n \geqslant 2$ being even or odd, as follows: If $n = 2N$ we set $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$, with $1 \leqslant j \leqslant N$. If $n = 2N-1$ we set $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ with $1 \leqslant j \leqslant N-1$ and $\rho_N = x_n$. In the first case set $d = N$ and in the second case set $d = N-1$. It is now seen that for $x \in \overline{\mathbb{X}}_n$ the vector $\rho = \rho(x)$ lies in the semi-annular domain $\overline{\mathbb{A}}_n \subset \mathbb{R}^N$ where $\mathbb{A}_n = \{\rho \in \mathbb{R}^d_+ : a < \|\rho\| < b\}$ when $n = 2N$ and $\mathbb{A}_n = \{\rho \in \mathbb{R}^d_+ \times \mathbb{R} : a < \|\rho\| < b\}$ when $n = 2N-1$ where we have set $\|\rho\| = (\rho_1^2 + \dots + \rho_N^2)^{1/2}$ for the 2-norm of the N-vector ρ. With this notation in place we require in (1.11) that $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$. As for boundary values and further structure of $\mathbf{Q}$, let us first write $\partial\mathbb{A}_n = (\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b \cup \Gamma_n$ where the three boundary segments of $\partial\mathbb{A}_n$ are defined as $\Gamma_n = \partial\mathbb{A}_n \setminus [(\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b]$ (the flat parts) and $(\partial\mathbb{A}_n)_a = \{\rho \in \partial\mathbb{A}_n : \|\rho\| = a\}$, $(\partial\mathbb{A}_n)_b = \{\rho \in \partial\mathbb{A}_n : \|\rho\| = b\}$. By consideration of 2-plane symmetries the matrix map $\mathbf{Q}$ is now confined to take values on the maximal torus $\mathbb{T}$ of $\mathbf{SO}(n)$ consisting of 2×2 block-diagonal rotation matrices, thus, in more explicit form

$$\mathbf{Q}(\rho_1, \dots, \rho_N) = \begin{cases} diag(\mathbf{R}[f_1], \dots, \mathbf{R}[f_d]), & n = 2d, \\ diag(\mathbf{R}[f_1], \dots, \mathbf{R}[f_d], 1), & n = 2d+1. \end{cases} \tag{1.12}$$

Here $\mathbf{R}$ is a 2×2 rotation matrix defined via (2.3) and the functions $f_j \in \mathscr{C}(\overline{\mathbb{A}}_n)$ for all $1 \leqslant j \leqslant d$ satisfy $f_j \equiv 0$ on $(\partial\mathbb{A}_n)_a$ and $f_j \equiv 2m_j\pi$ on $(\partial\mathbb{A}_n)_b$. Note that $x \in (\partial\mathbb{X}_n)_a = \{|x| = a\} \iff \rho(x) \in (\partial\mathbb{A}_n)_a$ and $x \in (\partial\mathbb{X}_n)_b = \{|x| = b\} \iff \rho(x) \in (\partial\mathbb{A}_n)_b$. The functions $f_j = f_j(\rho)$ and hence the map $\mathbf{Q} = \mathbf{Q}(\rho)$ are left free on the flat part of the boundary Γ_n. Again it can be seen that subject to the differentiability of the matrix field $\mathbf{Q}$, we have

$$\nabla\mathscr{R}_u = \theta, \qquad \nabla\mathscr{S}_u = \frac{1}{r}\mathbf{Q}(\mathbf{I}_n - \theta \otimes \theta) + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell}\theta \otimes \nabla\rho_\ell, \tag{1.13}$$

(with $\mathbf{Q}_{,\ell} = \partial\mathbf{Q}/\partial\rho_\ell$) and so referring to (1.8)-(1.12) and after a little more involved calculation it follows that $\det\nabla u = 1$.

Note that despite apparent similarities these two classes of maps are different in that in the first case (twists) the dependence of the twist path is on the radial variable $r = |x|$ only with no restriction on its range whereas in the second case (whirls) the dependence is on the 2-plane radial variables $\rho = (\rho_1, \dots, \rho_N)$ with the range restricted to a maximal torus. As such whirls are seen to have less symmetries than twists (*see* [25, 26] and [16, 17, 18, 19, 22]).

Our aim is to establish the existence of an infinitude of topologically twisting incompressible H-harmonic maps. Over the course of the paper it will become apparent that certain closed [*scaled*] geodesics of the compact Lie group $\mathbf{SO}(n)$ in the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ $(a \leqslant r \leqslant b)$ with $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$ will play a prominent role in relation to such solutions. The profile curve $\mathscr{H} \in \mathscr{C}^2[a,b]$ here is a solution to a two point boundary value problem [*cf.* (4.12)] and $\mathbf{H}$ is a fixed element of the Lie algebra $\mathfrak{so}(n)$. [4] In fact the profile curve $\mathscr{H}$ relates directly to the weight function H via the integral

$$\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{ds}{s^{n+1}H(s,s^2)}, \qquad a \leqslant r \leqslant b. \tag{1.14}$$

In Section 2 we arrive at a linear system for the *whirl* functions $f_1, \dots, f_d$ introduced in (1.12). It is proved in Theorem 3 that the *unique* solution to this system is such that each of the whirl functions depends on the modulus of the vector variable $\rho = (\rho_1, \dots, \rho_N)$ alone, *i.e.,* $f_\ell(\rho) = f_\ell(\|\rho\|)$ for all $1 \leqslant \ell \leqslant d$. By a further analysis of the PDE $\mathscr{L}_H[u] = \nabla\mathscr{P}$ we are then able to prove the following result that classifies all whirl solutions to the system (1.2) that in turn leads to the existence of an *infinite* family of incompressible H-harmonic maps of whirl type. (Except for (ii) in Part 1 where the only such solution is $u \equiv x$!)

THEOREM 1. *A whirl u associated with the matrix field $\mathbf{Q} \in \mathscr{C}^2(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$ satisfying $\mathbf{Q}(\rho) = \mathbf{I}_n$ for $\rho \in (\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b$ is a solution to the system* (1.2) *iff $\mathbf{Q} = \mathbf{Q}(\rho)$ is as described below.*

1. $(\,rH_r(r,r^2) + 2(n+1)H(r,r^2) + 4r^2H_s(r,r^2) \not\equiv 0$ *on* $]a,b[\,)$ *Here depending on the dimension n being even or odd we have*

   (i) *n even:* $\mathbf{Q}(\rho) = diag(\mathbf{R}[f_1(\rho)], \dots, \mathbf{R}[f_d(\rho)])$ $(\,\rho \in \overline{\mathbb{A}}_n\,)$ *where*

   $$f_\ell(\rho) = 2m_\ell\pi\mathscr{H}(||\rho||), \qquad 1 \leqslant \ell \leqslant d, \tag{1.15}$$

   with $m_1, \dots, m_d \in \mathbb{Z}$ satisfying $|m_1| = \ldots = |m_d|$.

   (ii) *n odd: $m_1 = \ldots = m_d = 0$ and therefore $\mathbf{Q} \equiv \mathbf{I}_n$.*

[4] We remark here that a solution to this system is $u \equiv x$. Indeed, upon substitution, (1.3) reduces to $\mathscr{L}_H[u \equiv x] = [H_r(r,r^2) + 2rH_s(r,r^2) - rnH_s(r,r^2)]\theta = \nabla\mathscr{P}$. The left-hand side here can be written as $s(r)\theta$ and as such is the gradient of some appropriate primitive function $s(r)\theta = \nabla S(|x|)$ that depends on the radial variable alone. The other equations in (1.2) are evidently satisfied by u.

2. $(\, rH_r(r,r^2)+2(n+1)H(r,r^2)+4r^2H_s(r,r^2) \equiv 0$ *on* $]a,b[\,)$ *Here we have* $\mathbf{Q}(\rho) = diag(\mathbf{R}[f_1(\rho)],\dots,\mathbf{R}[f_d(\rho)])$ $(\, \rho \in \overline{\mathbb{A}}_n\,)$ *when* n *is even* $(\, n=2d\,)$ *and likewise* $\mathbf{Q}(\rho) = diag(\mathbf{R}[f_1(\rho)],\dots,\mathbf{R}[f_d(\rho)],1)$ $(\, \rho \in \overline{\mathbb{A}}_n\,)$ *when* n *is odd* $(\, n=2d+1\,)$. *In either case* f_ℓ *is as in* (1.15) *for each* $1 \leqslant \ell \leqslant d$ *and with no restriction on the integers* $m_1,\dots,m_d$.

It is clear from the above theorem that the object $\mathscr{F}_H(r) := 2(n+1)H(r,r^2) + rH_r(r,r^2)+4r^2H_s(r,r^2)$ holds great influence on the structure of such solutions. This is a similar scenario when considering solutions of twist type. Indeed as will be justified in detail later in Section 3 in scrutinising the irrotational structure of the vector field $\mathscr{L}_H[u]$, we eventually arrive at the identity,

$$\mathrm{curl}\,\mathscr{L}_H[u] = [\nabla \mathscr{L}_H[u]] - [\nabla \mathscr{L}_H[u]]^t = \frac{\mathscr{F}_H(r)}{r^2}\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}x\otimes x - x\otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}x\right]. \tag{1.16}$$

If $\mathscr{F}_H \not\equiv 0$ then the irrotationality of $\mathscr{L}_H[u]$ and the solvability of $\mathscr{L}_H[u] = \nabla\mathscr{P}$ lead to an extreme dimensional dichotomy on solutions as reflected in Part 1 of the theorem. More interestingly when $\mathscr{F}_H(r) \equiv 0$, then $\mathscr{L}_H[u]$ being trivially irrotational, we obtain – notably in odd dimensions $n \geqslant 3$ – infinitely many non-trivial topologically twisting incompressible H-harmonic maps (compare Part 2 with (ii) in Part 1). Naturally the (r,s)-dependence in the weight function H here is crucial (*see* below). The counterpart of the above result for twists is now given in the following statement.

THEOREM 2. *A twist* u *with associated* $\mathbf{Q} \in \mathscr{C}([a,b],\mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[,\mathbf{SO}(n))$ *satisfying* $\mathbf{Q}(a) = \mathbf{I}_n$ *and* $\mathbf{Q}(b) = \mathbf{I}_n$ *is a solution to the system* (1.2) *iff* $\mathbf{Q}$ *is as described below.*

1. $(\, rH_r(r,r^2)+2(n+1)H(r,r^2)+4r^2H_s(r,r^2) \not\equiv 0$ *on* $]a,b[\,)$ *Here depending on the dimension* n *being even or odd we have*

   (i) *n even:* $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ $(\, a \leqslant r \leqslant b\,)$ *with* $\mathbf{H} = 2m\pi\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ *where* $\mathbf{P} \in \mathbf{O}(n)$, $m \in \mathbb{Z}$ *and* $\mathbf{J}_n = diag(\mathbf{J},\dots,\mathbf{J})$ *with* $\mathbf{J}$ *as in* (2.3).

   (ii) *n odd:* $\mathbf{H} \equiv 0$ *leading to* $\mathbf{Q} \equiv \mathbf{I}_n$. *Hence the identity map* $u \equiv x$ *is the only twisting solution of* (1.2)-(1.3).

2. $(\, rH_r(r,r^2)+2(n+1)H(r,r^2)+4r^2H_s(r,r^2) \equiv 0$ *on* $]a,b[\,)$ *Here we have* $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ $(\, a \leqslant r \leqslant b\,)$ *with* $\mathbf{H} = \mathbf{P}diag(2m_1\pi\mathbf{J},...,2m_d\pi\mathbf{J})\mathbf{P}^t$ *when* $n=2d$ *and* $\mathbf{H} = \mathbf{P}diag(2m_1\pi\mathbf{J},...,2m_d\pi\mathbf{J},0)\mathbf{P}^t$ *when* $n=2d+1$. *Moreover* $\mathbf{P} \in \mathbf{O}(n)$ *and* $m_1,\dots,m_d \in \mathbb{Z}$.

We close this introduction by giving, for the sake of illustration, a class of weights H that satisfy the condition $rH_r(r,r^2)+2(n+1)H(r,r^2)+4r^2H_s(r,r^2) \equiv 0$ and for which (1.2)-(1.3) admits an infinitude of non-trivial incompressible twisting H-harmonic maps regardless of n being even or odd. Towards this end let us take $H(r,s) = \mathsf{a}(r)\mathsf{b}(s) = r^\alpha s^\beta$ for real α, β, $a \leqslant r \leqslant b$ and $s>0$. Then $rH_r(r,r^2)+2(n+1)H(r,r^2)+4r^2H_s(r,r^2) \equiv 0 \iff r\dot{\mathsf{a}}\mathsf{b}+2(n+1)\mathsf{a}\mathsf{b}+4r^2\mathsf{a}\dot{\mathsf{b}} \equiv 0$ that is $\alpha+2(n+1)+4\beta = 0$

which then by substituting this into (1.1) yields

$$\mathbb{I}_\beta[u;\mathbb{X}_n] = \int_{\mathbb{X}_n} |\nabla u|^2\, d\mu(x,u) = \int_{\mathbb{X}_n} \frac{|u|^{2\beta}|\nabla u|^2}{|x|^{2(n+1)+4\beta}}\, dx. \tag{1.17}$$

Note that by linearity any finite sum $H(r,s) = \sum_j c_j r^{\alpha_j} s^{\beta_j}$ with $c_j > 0$ and $\alpha_j + 2(n+1) + 4\beta_j = 0$ still verifies $rH_r(r,r^2) + 2(n+1)H(r,r^2) + 4r^2 H_s(r,r^2) \equiv 0$. Of course these are by no means the only functions H satisfying the condition.

2. The restricted Euler-Lagrange system and its unique solvability

Recall that by definition a whirl is a self-map u with radial and spherical parts $\mathscr{R}_u = |x|$ and $\mathscr{S}_u = \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}$ respectively. The vector of 2-plane radial variables $\rho = (\rho_1, ..., \rho_N)$ and the semi-annular region $\mathbb{A}_n \subset \mathbb{R}^N$ and other related notions were defined earlier in Section 1 (*see* also Figure 1 below). Now assuming $\mathbf{Q} \in \mathscr{C}^1(\mathbb{A}_n, \mathbf{SO}(n)) \cap \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$ a basic calculation gives

$$\nabla u = \mathscr{R}_u \nabla \mathscr{S}_u + \mathscr{S}_u \otimes \nabla \mathscr{R}_u = \mathbf{Q} + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell} x \otimes \nabla \rho_\ell. \tag{2.1}$$

Here $\mathbf{Q}_{,\ell}$ denotes the first derivatives of $\mathbf{Q}$ with respect to ρ_ℓ whilst $\nabla \rho_\ell$ denotes the gradient of ρ_ℓ with respect to $x = (x_1, \dots, x_n)$. As a result we have

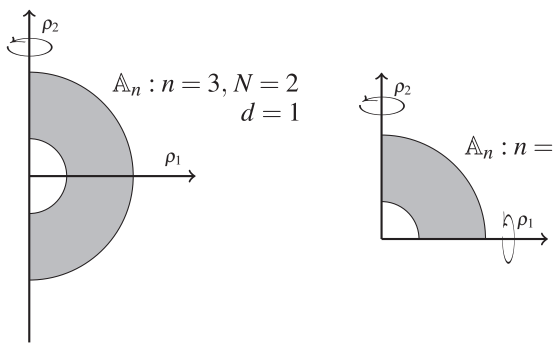

Figure 1: The contrasting symmetries in the semi-annular region $\mathbb{A}_n$ associated with $\mathbb{X}_n$ for n odd versus n even.

$$\begin{aligned} |\nabla u|^2 &= tr\left\{ \left(\mathbf{Q} + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell} x \otimes \nabla \rho_\ell \right) \left(\mathbf{Q}^t + \sum_{\ell=1}^{N} \nabla \rho_\ell \otimes \mathbf{Q}_{,\ell} x \right) \right\} \\ &= tr\left\{ \mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{Q} \nabla \rho_\ell \otimes \mathbf{Q}_{,\ell} x + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell} x \otimes \mathbf{Q} \nabla \rho_\ell + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell} x \otimes \mathbf{Q}_{,\ell} x \right\} \end{aligned}$$

$$= n + \sum_{\ell=1}^{N} |\mathbf{Q}_{,\ell} x|^2. \tag{2.2}$$

Here we have used $\langle \nabla \rho_j, \nabla \rho_k \rangle = \delta_{jk}$ and $\langle \mathbf{Q}^t \mathbf{Q}_{,\ell} x, \nabla \rho_k \rangle = 0$. Let us recall that throughout the paper we denote the $\mathbf{SO}(2)$ matrices $\mathbf{J}$ and $\mathbf{R}[t]$ by

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{R}[t] = \exp\{t\mathbf{J}\} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2.3}$$

Indeed to justify (2.2) note firstly that,

$$\mathbf{Q}^t \mathbf{Q}_{,\ell} = \begin{cases} \mathrm{diag}(\partial_\ell f_1 \mathbf{J}, \dots, \partial_\ell f_d \mathbf{J}), & \text{if } n = 2d, \\ \mathrm{diag}(\partial_\ell f_1 \mathbf{J}, \dots, \partial_\ell f_d \mathbf{J}, 0), & \text{if } n = 2d+1, \end{cases} \tag{2.4}$$

where $\partial_\ell f_k = \partial f_k / \partial \rho_\ell$. Let $y_k = (x_{2k-1}, x_{2k})$ for $1 \leqslant k \leqslant d$ regardless of $n = 2d$ or $n = 2d+1$ and $y_{d+1} = x_n$ when $n = 2d+1$. Then

$$\mathbf{Q}^t \mathbf{Q}_{,\ell} x = \begin{cases} (\partial_\ell f_1 \mathbf{J} y_1, \dots, \partial_\ell f_d \mathbf{J} y_d), & \text{if } n = 2d, \\ (\partial_\ell f_1 \mathbf{J} y_1, \dots, \partial_\ell f_d \mathbf{J} y_d, 0), & \text{if } n = 2d+1. \end{cases} \tag{2.5}$$

Furthermore from the definition of ρ_k it is clear that $\nabla \rho_k = (0, \dots, y_k/\rho_k, \dots, 0)$ and therefore $\langle \mathbf{Q}^t \mathbf{Q}_{,\ell} x, \nabla \rho_k \rangle = \langle \partial_\ell f_k \mathbf{J} y_k, y_k/\rho_k \rangle = 0$ in view of $\mathbf{J}$ being skew-symmetric. (Note that here there is no summation over $1 \leqslant \ell, k \leqslant N$ and the first equality excludes the relatively simpler case $k = N$ for n odd in which case the identity is trivially true.)

Let us also note that when $\mathbf{Q} \in \mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n)) \cap \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$ then upon taking the divergence of ∇u and using the identity $\nabla \rho_j \cdot \nabla \rho_k = \delta_{jk}$, we obtain

$$\Delta u = \sum_{\ell=1}^{N} \Big(\mathbf{Q}_{,\ell\ell} x + \Delta \rho_\ell \mathbf{Q}_{,\ell} x + 2 \mathbf{Q}_{,\ell} \nabla \rho_\ell \Big). \tag{2.6}$$

To verify that $u = r\mathbf{Q}(\rho_1, \dots, \rho_N)\theta$ satisfies the incompressibility constraint $\det \nabla u = 1$, using (2.1), we can write

$$\det \nabla u = \det \Big[\mathbf{Q} + \sum_{\ell=1}^{N} \mathbf{Q}_{,\ell} x \otimes \nabla \rho_\ell \Big] = \det \Big[\mathbf{I}_n + \sum_{\ell=1}^{N} \mathbf{Q}^t \mathbf{Q}_{,\ell} x \otimes \nabla \rho_\ell \Big]. \tag{2.7}$$

As seen earlier for any $1 \leqslant i, j \leqslant N$ we have $\langle \mathbf{Q}^t \mathbf{Q}_{,i} \theta, \nabla \rho_j \rangle = 0$. Now by Lemma 3.1 in [18] if a string of vectors $(a_k), (b_k)$ $(k = 1, \dots, N)$ in $\mathbb{R}^n$ satisfy $\langle a_i, b_j \rangle = 0$ for all i, j then $\det[\mathbf{I}_n + \sum_{j=1}^{N} a_j \otimes b_j] = 1$. An application of this identity with the choice of vectors $a_i = \mathbf{Q}^t \mathbf{Q}_{,i} x$ and $b_j = \nabla \rho_j$ immediately gives $\det \nabla u = 1$ as claimed.

The restricted system. Consider restricting the energy integral (1.1) to the class of admissible whirls u with $\mathbf{Q} = \mathbf{Q}(\rho)$ as in (1.12). Then $d\mu(x,u) = H(|x|, |u|^2) d\mathscr{L}^n = H(|x|, |x|^2) d\mathscr{L}^n$ and so by invoking (2.1)-(2.2) we can write

$$\mathbb{I}[u; \mathbb{X}_n] = \int_{\mathbb{X}_n} |\nabla u|^2 \, d\mu(x,u) = \int_{\mathbb{X}_n} H(|x|, |x|^2) \Big(n + \sum_{\ell=1}^{N} |\mathbf{Q}_{,\ell} x|^2 \Big) \, dx. \tag{2.8}$$

Recalling the block diagonal formulation of $\mathbf{Q}$ as given by (1.12) it is seen that

$$|\nabla u|^2 - |\nabla x|^2 = \sum_{\ell=1}^{N} |\mathbf{Q}_{,\ell} x|^2 = \sum_{\ell=1}^{N}\sum_{k=1}^{d} \rho_k^2 (f_{k,\ell})^2 = \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla f_\ell|^2. \tag{2.9}$$

Hence by substitution and a change of the variables of integration in (2.8) we can formulate $\mathbb{I}[u;\mathbb{X}_n] - \mathbb{I}[x;\mathbb{X}_n]$ as

$$\begin{aligned}\int_{\mathbb{X}_n} (|\nabla u|^2 - |\nabla x|^2)\, d\mu(x,u) &= \int_a^b \int_{\mathbb{S}^{n-1}} r^{n+1} H(r,r^2) \sum_{\ell=1}^{N} |\mathbf{Q}_{,\ell}\theta|^2 \, dr d\mathscr{H}^{n-1}(\theta) \\ &= (2\pi)^d \int_{\mathbb{A}_n} H(\|\rho\|,\|\rho\|^2) \sum_{\ell=1}^{d} \rho_\ell^2 |\nabla f_\ell|^2 \prod_{j=1}^{d} \rho_j \, d\rho \\ &=: (2\pi)^d \sum_{\ell=1}^{d} \mathbb{J}_\ell[f_\ell;\mathbb{A}_n].\end{aligned} \tag{2.10}$$

As for the Jacobian of this coordinate transformation note that we hereafter set

$$\mathscr{J}(\rho) = \mathscr{J}(\rho_1,\dots,\rho_N) = \prod_{j=1}^{d} \rho_j, \tag{2.11}$$

and so when $n = 2d$ (with $N = d$) this product contains all $\rho_1,\dots,\rho_N$, whereas when $n = 2d+1$ (with $N = d+1$) the product contains $\rho_1,\dots,\rho_{N-1}$. Referring to (2.10) we have denoted the restricted energy $\mathbb{J}_\ell$ (with $1 \leqslant \ell \leqslant d$) by

$$\mathbb{J}_\ell[f;\mathbb{A}_n] := \int_{\mathbb{A}_n} |\nabla f|^2 \omega_\ell(\rho)\, d\rho, \qquad \omega_\ell(\rho) = \rho_\ell^2 H(\|\rho\|,\|\rho\|^2) \mathscr{J}(\rho). \tag{2.12}$$

Evidently $\mathbb{J}_\ell$ $(1 \leqslant \ell \leqslant d)$ is a weighted Dirichlet energy (here unconstrained though) and as such will be considered over the space of admissible functions $f = f(\rho) = f(\rho_1,...,\rho_N)$ in the space

$$\mathscr{B}(\mathbb{A}_n) = \bigcup_{m\in\mathbb{Z}} \mathscr{B}_m(\mathbb{A}_n), \tag{2.13}$$

where $\mathscr{B}_m(\mathbb{A}_n) = \{f \in \mathscr{W}^{1,2}(\mathbb{A}_n) : f = 0 \text{ on } (\partial\mathbb{A}_n)_a,\ f = 2m\pi \text{ on } (\partial\mathbb{A}_n)_b\}$. With the aim of finding solutions to the system (1.2) in the form of whirl maps we proceed first on to extremising the restricted energy $\mathbb{J}_\ell$ over the space $\mathscr{B}_m(\mathbb{A}_n)$. Now the Euler-Lagrange equation for $\mathbb{J}_\ell$ over $\mathscr{B}_m(\mathbb{A}_n)$ is seen to be (with $1 \leqslant \ell \leqslant d$, $m \in \mathbb{Z}$):

$$\mathbf{EL}[f;\mathbb{J}_\ell,\mathbb{A}_n] = \begin{cases} \operatorname{div}[\omega_\ell(\rho)\nabla f] = 0 & \text{in } \mathbb{A}_n, \\ f = 0 & \text{on } (\partial\mathbb{A}_n)_a, \\ f = 2m\pi & \text{on } (\partial\mathbb{A}_n)_b, \\ \omega_\ell(\rho)\partial_\nu f = 0 & \text{on } \Gamma_n. \end{cases} \tag{2.14}$$

Here $\omega_\ell(\rho) = H(\|\rho\|,\|\rho\|^2)\rho_\ell^2 \mathscr{J}(\rho)$ is a strictly positive weight (for $\rho \in \mathbb{A}_n$) and $\partial_\nu f = \nabla f \cdot \nu$ where ν is the unit outward normal field on Γ_n. We next aim to show that this system has the unique solution $f = f(\rho;m) = 2m\pi\mathscr{H}(\|\rho\|)$ with the choice of $\mathscr{H} = \mathscr{H}(r) \in \mathscr{C}^2[a,b]$, precisely as given by (1.14).

Towards this end it is firstly seen that the boundary conditions on f are satisfied by virtue of $\mathscr{H}(a) = 0$ and $\mathscr{H}(b) = 1$ along with $\omega_\ell = 0$ on Γ_n. Next proceeding on to the PDE on the first line, by direct differentiation and reference to (1.14) we have,

$$\frac{\partial f}{\partial \rho_i} = 2m\pi \frac{\dot{\mathsf{H}}(r)}{\mathsf{H}(b)} \frac{\rho_i}{r} = \frac{2m\pi}{\mathsf{H}(b)} \frac{\rho_i}{r^{n+2} H(r,r^2)}, \qquad 1 \leqslant i \leqslant N. \tag{2.15}$$

Now specialising first to even dimensions $n = 2d$, $N = d$, a straightforward calculation gives,

$$\begin{aligned} &\operatorname{div}[\omega_\ell(\rho)\nabla f] \\ =&\operatorname{div}\left[H(r,r^2)\rho_\ell^2 \mathscr{J}(\rho)\nabla f\right] = \sum_{i=1}^{N} \frac{\partial}{\partial \rho_i} \frac{2m\pi}{\mathsf{H}(b)} \left(H(r,r^2) \frac{\rho_i \rho_\ell^2 \mathscr{J}(\rho)}{r^{n+2} H(r,r^2)} \right) \\ =&\frac{2m\pi}{\mathsf{H}(b)} \sum_{i=1}^{d} \left(\frac{\rho_\ell^2 \mathscr{J}(\rho)}{r^{n+2}} - (n+2)\frac{\rho_i^2\rho_\ell^2}{r^{n+4}} \mathscr{J}(\rho) + 2\frac{\rho_i\rho_\ell\delta_{i\ell}}{r^{n+2}} \mathscr{J}(\rho) + \frac{\rho_i\rho_\ell^2}{r^{n+2}} \frac{\mathscr{J}(\rho)}{\rho_i} \right) \\ =&\frac{2m\pi}{\mathsf{H}(b)} \frac{\rho_\ell \mathscr{J}(\rho)}{r^{n+2}} \Big(d\rho_\ell - (2d+2)\rho_\ell + 2\rho_\ell + d\rho_\ell \Big) = 0. \end{aligned} \tag{2.16}$$

Next for $n = 2d+1$, $N = d+1$ we proceed similarly but recall that $\rho_N = x_n$. For the first $\rho_1, \dots, \rho_d$ terms we have,

$$\frac{2m\pi}{\mathsf{H}(b)} \sum_{i=1}^{d} \frac{\partial}{\partial \rho_i} \left(\frac{\rho_i \rho_\ell^2}{r^{n+2}} \mathscr{J}(\rho) \right) = \frac{2m\pi}{\mathsf{H}(b)} \frac{\rho_\ell \mathscr{J}(\rho)}{r^{n+2}} \left(d\rho_\ell - \frac{(2d+3)}{r^2} \rho_\ell \sum_{i=1}^{d} \rho_i^2 + 2\rho_\ell + d\rho_\ell \right). \tag{2.17}$$

To this we add the N^{th} term in the divergence sum, which is then seen to be

$$\frac{2m\pi}{\mathsf{H}(b)} \frac{\partial}{\partial \rho_N} \left(\frac{\rho_N \rho_\ell^2}{r^{n+2}} \mathscr{J}(\rho) \right) = \frac{2m\pi}{\mathsf{H}(b)} \frac{\rho_\ell \mathscr{J}(\rho)}{r^{n+2}} \left(\rho_\ell - \frac{(2d+3)}{r^2} \rho_\ell \rho_N^2 \right). \tag{2.18}$$

Coupling this latter expression with the earlier sum (2.17) therefore gives

$$\begin{aligned} \operatorname{div}[\omega_\ell(\rho)\nabla f] = \operatorname{div}\left[H(r,r^2)\rho_\ell^2 \mathscr{J}(\rho)\nabla f\right] &= \frac{2m\pi}{\mathsf{H}(b)} \sum_{i=1}^{N} \frac{\partial}{\partial \rho_i} \left(\frac{\rho_i \rho_\ell^2}{r^{n+2}} \mathscr{J}(\rho) \right) \\ &= \frac{2m\pi}{\mathsf{H}(b)} \frac{\rho_\ell \mathscr{J}(\rho)}{r^{n+2}} \Big(d\rho_\ell - (2d+3)\rho_\ell + 2\rho_\ell + (d+1)\rho_\ell \Big) = 0. \end{aligned} \tag{2.19}$$

An energy argument combined with the above now implies the uniqueness of the proposed solution $f(\rho;m)$ to the boundary value problem (2.14).

THEOREM 3. *Let $f(\rho,m) = 2m\pi\mathscr{H}(||\rho||)$ with $\mathscr{H} = \mathscr{H}(r)$ as in* (1.14)*, $m \in \mathbb{Z}$ and let $1 \leqslant \ell \leqslant d$. Then $f = f_\ell$ in $\mathscr{C}^2(\overline{\mathbb{A}}_n)$ is the* unique *solution to the system* (2.14) *and the unique minimiser of the restricted energy $\mathbb{J}_\ell[f;\mathbb{A}_n]$ over $\mathscr{B}_m(\mathbb{A}_n)$.*

Proof. That $f = f(\rho;m)$ solves (2.14) has already been established so it remains to prove the uniqueness statement in the theorem. Towards this end fix ℓ, m and suppose for the sake of a contradiction that f^1, f^2 are solutions and put $f = f^2 - f^1$. Then f is a solution to (2.14) with zero right-hand side, *i.e.*, $f \equiv 0$ on $(\partial\mathbb{A}_n)_a \cup (\partial\mathbb{A}_n)_b$ and $H(\|\rho\|, \|\rho\|^2)\rho_\ell^2 \mathscr{J}(\rho)\partial_\nu f \equiv 0$ on Γ_n. The divergence theorem along with the PDE satisfied by f then gives

$$\int_{\mathbb{A}_n} |\nabla f|^2 \omega_\ell(\rho)\, d\rho = \int_{\Gamma_n} \partial_\nu f \omega_\ell(\rho)\, d\mathscr{H}^{n-1} = 0.$$

Now since we have $H, \rho_j > 0$ in $\mathbb{A}_n$ for all $1 \leqslant j \leqslant d$, it follows that $f \equiv 0$ by noting the connectedness of $\mathbb{A}_n$ and the zero boundary conditions on f. Therefore $f^1 = f^2$ and so the uniqueness follows. Next for any $g \in \mathscr{B}_m(\mathbb{A}_n)$ by writing $\phi = g - f$ and invoking (2.14) we have

$$\mathbb{J}_\ell[g;\mathbb{A}_n] - \mathbb{J}_\ell[f;\mathbb{A}_n] = \int_{\mathbb{A}_n} (|\nabla g|^2 - |\nabla f|^2)\omega_\ell(\rho)\, d\rho \geqslant \int_{\mathbb{A}_n} |\nabla \phi|^2 \omega_\ell(\rho)\, d\rho,$$

justifying the unique minimality of $f = f_\ell(\rho;m)$ as required. $\square$

3. Whirls as solutions to the system (1.2) and the proof of Theorem 1

Our attention now shifts to the system (1.2) and the PDE $\mathscr{L}_H[u] = \nabla\mathscr{P}$. Here the action of $\mathscr{L}_H$ on a whirl u with $\mathbf{Q} \in \mathscr{C}^2(\mathbb{A}_n, \mathbf{SO}(n)) \cap \mathscr{C}(\overline{\mathbb{A}}_n, \mathbf{SO}(n))$ can be written

$$\begin{aligned}\mathscr{L}_H[u] = \left(\mathbf{Q}^t + \sum_{\ell=1}^N \nabla\rho_\ell \otimes \mathbf{Q}_{,\ell}x\right) \Bigg\{ [H_r + 2rH_s]\left(\mathbf{Q} + \sum_{\ell=1}^N \mathbf{Q}_{,\ell}x \otimes \nabla\rho_\ell\right)\theta + \\ + H\sum_{\ell=1}^N \left[\mathbf{Q}_{,\ell\ell}x + \Delta\rho_\ell \mathbf{Q}_{,\ell}x + 2\mathbf{Q}_{,\ell}\nabla\rho_\ell\right] - rH_s\left(n + \sum_{\ell=1}^N |\mathbf{Q}_{,\ell}x|^2\right)\mathbf{Q}\theta \Bigg\} \quad (3.1)\end{aligned}$$

where $H = H(r, r^2)$, $H_r = H_r(r, r^2)$ and $H_s = H_s(r, r^2)$. The above formulation follows from (1.3) by substituting for ∇u from (2.1), for $|\nabla u|^2$ from (2.2) and for Δu from (2.6).

Motivated by the results in the previous section we now specialise to the case $f_\ell(\rho;m_\ell) = 2m_\ell\pi\mathscr{H}(||\rho||)$ for $1 \leqslant \ell \leqslant d$ with $\mathscr{H} \in \mathscr{C}^2[a,b]$ as in (1.14). First let us pause briefly to re-examine the identities obtained earlier for whirls, now for the case where, with a slight abuse of notation, $\mathbf{Q} = \mathbf{Q}(||\rho||)$. Beginning with the gradient and noting that $r = \|\rho\| = (\rho_1^2 + ... + \rho_N^2)^{1/2}$ we have $\mathbf{Q}_{,\ell} = \rho_\ell / r\dot{\mathbf{Q}}(r)$ (with $\dot{\mathbf{Q}} = d\mathbf{Q}/dr$) by a straightforward differentiation. Therefore by recalling (2.1) we have

$$\nabla u = \mathbf{Q} + \dot{\mathbf{Q}}\theta \otimes \sum_{\ell=1}^N \rho_\ell \nabla\rho_\ell = \mathbf{Q} + \dot{\mathbf{Q}}\theta \otimes x = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta \quad (3.2)$$

by virtue of $\sum_{\ell=1}^N \rho_\ell \nabla\rho_\ell = \nabla||\rho||^2/2 = x$. In particular $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2$. Now for the Laplacian Δu we first note that $\Delta\rho_\ell = 1/\rho_\ell$ except for n odd and $\ell = N$ where

$\Delta\rho_N = 0$ whilst $\mathbf{Q}_{,\ell\ell} = \rho_\ell^2/r^2\ddot{\mathbf{Q}} + (r^2-\rho_\ell^2)/r^3\dot{\mathbf{Q}}$. Recalling (2.6) it is therefore plain that

$$\begin{aligned}\Delta u &= \sum_{\ell=1}^{N}\left(\mathbf{Q}_{,\ell\ell}x + \Delta\rho_\ell \mathbf{Q}_{,\ell}x + 2\mathbf{Q}_{,\ell}\nabla\rho_\ell\right)\\ &= \sum_{\ell=1}^{N}\left\{\left(\frac{\rho_\ell^2}{r^2}\ddot{\mathbf{Q}} + \frac{r^2-\rho_\ell^2}{r^3}\dot{\mathbf{Q}}\right)x + \frac{\rho_\ell}{r}\Delta\rho_\ell\dot{\mathbf{Q}}x + 2\frac{\rho_\ell}{r}\dot{\mathbf{Q}}\nabla\rho_\ell\right\}. \end{aligned} \tag{3.3}$$

This is therefore seen to give $\Delta u = r\ddot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + N\dot{\mathbf{Q}}\theta + 2\dot{\mathbf{Q}}\theta$ for n even and $\Delta u = r\ddot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + (N-1)\dot{\mathbf{Q}}\theta + 2\dot{\mathbf{Q}}\theta$ for n odd. Thus in conclusion $\Delta u = r\ddot{\mathbf{Q}}\theta + (n+1)\dot{\mathbf{Q}}\theta$ as $N = n/2$ when n is even and $N = (n+1)/2$ when n is odd. With these basic identities at hand the action $\mathscr{L}_H[u]$ can be written as

$$\begin{aligned}\mathscr{L}_H[u] =& (\mathbf{Q}^t + r\theta\otimes\dot{\mathbf{Q}}\theta)\Big\{[H_r(r,r^2) + 2rH_s(r,r^2)](\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta)\\ &+ H(r,r^2)\left[r\ddot{\mathbf{Q}} + (n+1)\dot{\mathbf{Q}}\right]\theta - rH_s(r,r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\theta\Big\}. \end{aligned} \tag{3.4}$$

Before proceeding with the proof of the main theorem we present the following technical lemma on the irrotationality of certain $\mathscr{C}^1$ vector fields generated by skew-symmetric matrices and the implication of this on the vector field being a gradient field.

LEMMA 1. *Let* $\mathscr{A} = \mathscr{A}(r,z)$, $\mathscr{B} = \mathscr{B}(r,z) \in \mathscr{C}^1(]a,b[\times\mathbb{R},\mathbb{R})$ *and let* $\mathbf{H}$ *be an* $n\times n$ *skew-symmetric matrix written* $\mathbf{H} = \mathbf{P}diag(h_1\mathbf{J},\dots,h_k\mathbf{J})\mathbf{P}^t$ *when* $n=2k$ *and* $\mathbf{H} = \mathbf{P}diag(h_1\mathbf{J},\dots,h_{k-1}\mathbf{J},h_k)\mathbf{P}^t$ *when* $n = 2k-1$. *Here* $\mathbf{P}\in\mathbf{O}(n)$ *is fixed,* $(h_j : 1\leqslant j\leqslant k)\subset\mathbb{R}$, *and* $\mathbf{J}$ *is given by* (2.3). *Consider the vector field* U *defined by*

$$U(x) = \mathscr{A}(|x|,|\mathbf{H}x|^2)x + \mathscr{B}(|x|,|\mathbf{H}x|^2)\mathbf{H}^2x, \qquad x\in\mathbb{X}_n, \tag{3.5}$$

and let $\Delta(r,z) := 2\mathscr{A}_z + \mathscr{B}_r/r$ *where* $\mathscr{A}_z$ *denotes the derivative of* $\mathscr{A}=\mathscr{A}(r,z)$ *in the second variable and* $\mathscr{B}_r$ *denotes the derivative of* $\mathscr{B}=\mathscr{B}(r,z)$ *in the first variable and set* $r=|x|$, $z=|\mathbf{H}x|^2$. *Then the following hold:*

- *If* $\Delta\not\equiv 0$ *in* $\mathbb{X}_n$, *then*

$$\mathrm{curl}\,U \equiv 0 \text{ in } \mathbb{X}_n \iff |h_1| = \ldots = |h_k| := h \iff \mathbf{H}^2 = -h^2\mathbf{I}_n. \tag{3.6}$$

- *If* $\Delta\equiv 0$ *in* $\mathbb{X}_n$ *then* $\mathrm{curl}\,U\equiv 0$ *in* $\mathbb{X}_n$ *with no further restriction on* $\mathbf{H}$.

In either case the vector field U *is a gradient field in* $\mathbb{X}_n$.

Before presenting the proof we note that firstly every skew-symmetric matrix can be orthogonally diagonalised and so the description of $\mathbf{H}$ above is precisely referring to this representation. Next the numbers $(\pm\sqrt{-1}h_j : 1\leqslant j\leqslant k)$ when $n=2k$ and $(\pm\sqrt{-1}h_j, h_k = 0 : 1\leqslant j\leqslant k-1)$ when $n = 2k-1$ are the eigenvalues of $\mathbf{H}$.

Proof. Let us begin by calculating $\operatorname{curl} U = [\nabla U] - [\nabla U]^t$. To this end, noting the skew-symmetry of $\mathbf{H}$, the symmetry of $\mathbf{H}^2$ and $\mathbf{H}^2 = -\mathbf{H}^t\mathbf{H}$, we can write upon suppressing the arguments in $\mathscr{A}, \mathscr{B}$ and their derivatives in the interest of brevity that

$$\nabla U = \frac{\mathscr{A}_r}{r} x \otimes x - 2\mathscr{A}_z x \otimes \mathbf{H}^2 x + \mathscr{A}\mathbf{I}_n + \frac{\mathscr{B}_r}{r}\mathbf{H}^2 x \otimes x - 2\mathscr{B}_z \mathbf{H}^2 x \otimes \mathbf{H}^2 x + \mathscr{B}\mathbf{H}^2.$$

As a result it is then clear that

$$\begin{aligned}[\nabla U]^t =& \frac{\mathscr{A}_r}{r} x \otimes x - 2\mathscr{A}_z \mathbf{H}^2 x \otimes x + \mathscr{A}\mathbf{I}_n + \frac{\mathscr{B}_r}{r} x \otimes \mathbf{H}^2 x \\ & - 2\mathscr{B}_z \mathbf{H}^2 x \otimes \mathbf{H}^2 x + \mathscr{B}\mathbf{H}^2.\end{aligned}$$

Therefore after taking into account the necessary cancellations we obtain

$$\begin{aligned}\operatorname{curl} U &= \left(2\mathscr{A}_z(|x|, |\mathbf{H}x|^2) + \frac{\mathscr{B}_r(|x|, |\mathbf{H}x|^2)}{r}\right)[\mathbf{H}^2 x \otimes x - x \otimes \mathbf{H}^2 x] \\ &= \Delta(|x|, |\mathbf{H}x|^2)[\mathbf{H}^2 x \otimes x - x \otimes \mathbf{H}^2 x]. \end{aligned} \tag{3.7}$$

Let $\mathbf{D} = diag(h_1\mathbf{J}, \dots, h_k\mathbf{J})$ when $n = 2k$ and $\mathbf{D} = diag(h_1\mathbf{J}, \dots, h_{k-1}\mathbf{J}, h_k)$ when $n = 2k-1$. Then $\mathbf{H} = \mathbf{PDP}^t$. As $\mathbb{X}_n$ is rotationally invariant the change of variables $y = \mathbf{P}^t x$ in (3.5) leaves $\mathbb{X}_n$ fixed and relates the $\mathscr{C}^1$ vector field U with V via $U(x) = \mathbf{P}[\mathscr{A}(|y|, |\mathbf{D}y|^2)y + \mathscr{B}(|y|, |\mathbf{D}y|^2)\mathbf{D}^2 y] =: \mathbf{P}V(y)$. In a similar way the same change of variables in (3.7) gives the relation

$$\operatorname{curl} U = \mathbf{P}\Delta(|y|, |\mathbf{D}y|^2)[\mathbf{D}^2 y \otimes y - y \otimes \mathbf{D}^2 y]\mathbf{P}^t = \mathbf{P}[\operatorname{curl} V]\mathbf{P}^t. \tag{3.8}$$

Thus by virtue of $\Delta(|x|, |\mathbf{H}x|^2) = \Delta(|y|, |\mathbf{D}y|^2)$ it is easily seen that it suffices to justify the assertion of the lemma for when $\mathbf{P} = \mathbf{I}_n$ as then $\mathbf{D}^2 = -h^2\mathbf{I}_n$ *iff* $\mathbf{H}^2 = -h^2\mathbf{I}_n$ and $V(y) = \nabla\phi(y)$ *iff* $U(x) = \mathbf{P}\nabla\phi(y) = \mathbf{P}\nabla\phi(\mathbf{P}^t x) = \nabla\phi(\mathbf{P}^t x)$. In the rest of the proof we thus confine to $\mathbf{P} = \mathbf{I}_n$. Indeed here we can write $\mathbf{H}^2 = -diag(h_1^2\mathbf{I}_2, \dots, h_k^2\mathbf{I}_2)$ when $n = 2k$ and $\mathbf{H}^2 = -diag(h_1^2\mathbf{I}_2, \dots, h_{k-1}^2\mathbf{I}_2, h_k^2)$ when $n = 2k-1$. Let $s(l) = [(l+1)/2]$ for $1 \leqslant l \leqslant n$. Then (3.7) can be written component-wise with $1 \leqslant i, j \leqslant n$ as

$$[\operatorname{curl} U]_{ij} = -\Delta(|x|, |\mathbf{H}x|^2)\left(h_{s(i)}^2 - h_{s(j)}^2\right)x_i x_j. \tag{3.9}$$

From this it follows that if $\Delta \not\equiv 0$ in $\mathbb{X}_n$ then $\operatorname{curl} U \equiv 0$ in $\mathbb{X}_n$ if and only if $h_1^2 = \dots = h_k^2$. (Note that firstly Δ is a continuous function of x and so if it does not vanish at a point then it does not vanishes in a neighbourhood of the point and secondly that the factors $x_i x_j$ vanish only on the coordinate hyperplanes.) Likewise if $\Delta \equiv 0$ in $\mathbb{X}_n$ then $\operatorname{curl} U \equiv 0$ in $\mathbb{X}_n$ with no impositions to be made on $h_1, \dots, h_k$. This proves the first part of the lemma.

We now prove that in either case U is a gradient field. First suppose that $\Delta \not\equiv 0$ and $h_1^2 = \dots = h_k^2$. In this case $U(x) = [\mathscr{A}(r, h^2r^2) - h^2\mathscr{B}(r, h^2r^2)]x = s(r)x$ and so there clearly exists a (radial) $\phi \in \mathscr{C}^2(\mathbb{X}_n)$ such that $U = \nabla\phi$. Next suppose that $\Delta \equiv 0$. We claim that $U(x) = \nabla f(|x|, |\mathbf{H}x|^2)$ for a suitable choice of $f = f(r,z)$ of class $\mathscr{C}^2$. Indeed assuming this were the case, by direct differentiation, we would have,

$$\nabla f(|x|, |\mathbf{H}x|^2) = f_r(r, |\mathbf{H}x|^2)\theta - 2rf_z(r, |\mathbf{H}x|^2)\mathbf{H}^2\theta$$

$$= \mathscr{A}(r, |\mathbf{H}x|^2)x + \mathscr{B}(r, |\mathbf{H}x|^2)\mathbf{H}^2 x = U(x) \tag{3.10}$$

provided that we set $f_r(r,z) = r\mathscr{A}(r,z)$ and $f_z(r,z) = -\mathscr{B}(r,z)/2$. Now let $\mathscr{R} = \{(r,z) : r = |x|,\ z = |\mathbf{H}x|^2 \text{ with } x \in \mathbb{X}_n\}$. Denoting by $\underline{\lambda}, \overline{\lambda} \geqslant 0$ the minimum and maximum eigenvalues $h_1^2, \ldots, h_k^2$ of the diagonal matrix $\mathbf{H}^t\mathbf{H}$ respectively it is easily seen that $\mathscr{R} = \{(r,z) : a < r < b, 0 \leqslant \underline{\lambda} r^2 \leqslant z \leqslant \overline{\lambda} r^2\}$. Since $\Delta \equiv 0$ we have $\partial_z f_r(r,z) - \partial_r f_z(r,z) = r\mathscr{A}_z(r,z) + \mathscr{B}_r(r,z)/2 \equiv 0$ in $\mathscr{R}$. As a result the 1-form $\omega = r\mathscr{A}(r,z)\,dr - \mathscr{B}(r,z)/2\,dz$ is closed in $\mathscr{R}$ and hence exact in view of $\mathscr{R}$ being simply-connected. Thus $\omega = df$ for a function (a 0-form) $f = f(r,z)$ of class $\mathscr{C}^2$. To describe f more specifically pick a base point (r_0, z_0) in $\mathscr{R}$ and let γ be any piecewise continuously differentiable Jordan curve in $\mathscr{R}$ connecting (r_0, z_0) to (r,z) and set

$$f(r,z) = \int_\gamma \omega = \int_\gamma r\mathscr{A}(r,z)\,dr - \mathscr{B}(r,z)/2\,dz, \qquad (r,z) \in \mathscr{R}. \tag{3.11}$$

The integral is seen to be independent of the choice of γ and hence well-defined. The function f is of class $\mathscr{C}^2$ in the interior of $\mathscr{R}$ with continuously differentiable tangential gradients on the upper and lower boundary curves of $\mathscr{R}$. One can thus verify that (3.10) holds (both for $(r,z) = (|x|, |\mathbf{H}x|^2)$ in the interior of $\mathscr{R}$ and the upper and lower boundary curves). Thus by setting $\phi(x) = f(|x|, |\mathbf{H}x|^2)$ we have $U = \nabla\phi$. This completes the proof. $\square$

Proof of Theorem 1. : From the extremality analysis and explicit description of $f_1, \ldots, f_d$ in Theorem 3 it suffices to confine to whirls u with $\mathbf{Q}(\rho) = \exp\{\mathscr{H}(||\rho||)\mathbf{H}\}$ where $\mathscr{H}$ is as given by (1.14), $\mathbf{H} = diag(2m_1\pi\mathbf{J}, \ldots, 2m_d\pi\mathbf{J})$ for $n = 2d$ even and $\mathbf{H} = diag(2m_1\pi\mathbf{J}, \ldots, 2m_d\pi\mathbf{J}, 0)$ for $n = 2d+1$ odd. (Recall that $\mathbf{J} \in \mathbf{SO}(2)$ is as in (2.3).) Starting from the formulation of $\mathscr{L}_H[u]$ in (3.4) it then follows that (after a rearrangement)

$$\begin{aligned} \mathscr{L}_H[u] = &\nabla H(|x|, |x|^2) - nrH_s(r, r^2)\theta \\ &+ \left[r^2 H_r(r, r^2) + r^3 H_s(r, r^2) + r(n+1)H(r, r^2) \right] \dot{\mathscr{H}}^2 |\mathbf{H}\theta|^2 \theta \\ &+ r^2 H(r, r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}} |\mathbf{H}\theta|^2 \theta + rH(r, r^2)\dot{\mathscr{H}}^2 \mathbf{H}^2 \theta. \end{aligned} \tag{3.12}$$

Given that the first two terms on the right in (3.12) are gradient fields (and in particular irrotational) we henceforth set $U(x) := \mathscr{L}_H[u] - \nabla H + nrH_s\theta$. From here we can use Lemma 1 above, specifically, with $\mathscr{B}(r,z) = H(r, r^2)\dot{\mathscr{H}}^2$ and $\mathscr{A}(r,z) = [H_r(r, r^2)/r + H_s(r, r^2) + (n+1)H(r, r^2)/r^2]\dot{\mathscr{H}}^2 z + H(r, r^2)/r\dot{\mathscr{H}}\ddot{\mathscr{H}} z$. An initial inspection shows that with $\Delta = 2\mathscr{A}_z(r,z) + \mathscr{B}_r(r,z)/r$ we have

$$\begin{aligned} r^2\Delta &= [3rH_r(r, r^2) + 4r^2 H_s(r, r^2) + 2(n+1)H(r, r^2)]\dot{\mathscr{H}}^2 + 4rH(r, r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}} \\ &= -\dot{\mathscr{H}}^2 \left[2(n+1)H(r, r^2) + rH_r(r, r^2) + 4r^2 H_s(r, r^2) \right] \end{aligned} \tag{3.13}$$

where at this stage in proceedings we have used the fact that $\mathscr{H}$ given by (1.14) solves the ODE $d/dr[r^{n+1}H(r, r^2)\dot{\mathscr{H}}(r)] = 0$ to deduce the second line. Hence it follows that $\Delta = 2\mathscr{A}_z(r,z) + \mathscr{B}_r(r,z)/r = -\dot{\mathscr{H}}^2 \mathscr{F}_H(r)/r^2$.

An appeal to Lemma 1 leads to the conclusion that if $\mathscr{F}_H(r) \not\equiv 0$ then $\operatorname{curl} \mathscr{L}_H[u] = 0 \iff \mathbf{H}^2 = -c^2\mathbf{I}_n \iff |f_1|^2 = \ldots = |f_d|^2$ when $n = 2d$ is even and $|f_1|^2 = \ldots = |f_d|^2 = 0$ when $n = 2d+1$ is odd. As such this gives $m_\ell = 0$ for all $1 \leqslant \ell \leqslant d$ when $n = 2d+1$ and $|m_1| = \ldots = |m_d| =: |m|$ when $n = 2d$ and so $f_\ell \in \{\pm 2m\pi \mathscr{H}(||\rho||)\}$ for all $1 \leqslant \ell \leqslant d$. If $\mathscr{F}_H(r) \equiv 0$ we have, again by Lemma 1, that $\mathscr{L}_H[u]$ is irrotational and in fact a gradient field with no restriction on the integers $m_1, \ldots, m_d$. $\square$

4. Extremality of twists and closed geodesics on $\mathbf{SO}(n)$

In this section we take a closer look at twist maps and present some consequences of extremality. Recall that by definition a twist is a self-map u whose radial and spherical parts have the forms $\mathscr{R}_u = |x|$ and $\mathscr{S}_u = \mathbf{Q}(|x|)x|x|^{-1}$ respectively. Assuming $\mathbf{Q} \in \mathscr{C}^1(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$ we have $\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$ and thus $\det \nabla u = \det(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta) = 1$. Moreover $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2$. If additionally $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$ then $\Delta u = [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta$. As a result for the action $\mathscr{L}_H[u]$ as given by (1.3) we can write

$$\begin{aligned}\mathscr{L}_H[u] =&(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\{[H_r(r,r^2) + 2rH_s(r,r^2)](\mathbf{Q} + r\dot{\mathbf{Q}})\\ &+ H(r,r^2)[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}] - rH_s(r,r^2)(n + r^2|\dot{\mathbf{Q}}\theta|^2)\mathbf{Q}\}\theta.\end{aligned} \tag{4.1}$$

Expanding this further and introducing the skew-symmetric matrix field $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$ we can write for $a < r < b$

$$\begin{aligned}\mathscr{L}_H[u] =&[H_r(r,r^2) + 2rH_s(r,r^2)](\theta + r\mathbf{A}\theta + r^2|\mathbf{A}\theta|^2\theta)\\ &+ H(r,r^2)[(n+1)\mathbf{A}\theta + r(\dot{\mathbf{A}} + \mathbf{A}^2)\theta\\ &+ r(n+1)|\mathbf{A}\theta|^2\theta + r^2\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\theta] - rH_s(r,r^2)(n + r^2|\mathbf{A}\theta|^2)\theta.\end{aligned} \tag{4.2}$$

The above description follows upon noting $|\dot{\mathbf{Q}}\theta|^2 = |\mathbf{A}\theta|^2$, $\mathbf{Q}^t\ddot{\mathbf{Q}} = \dot{\mathbf{A}} + \mathbf{A}^2$ and $\langle \dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle = \langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \mathbf{Q}^t\ddot{\mathbf{Q}}\theta\rangle = \langle \mathbf{A}\theta, (\dot{\mathbf{A}} + \mathbf{A}^2)\theta\rangle = \langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle + \langle \mathbf{A}\theta, \mathbf{A}^2\theta\rangle = \langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle$ in view of $\mathbf{A}$ being skew-symmetric. Now a straightforward inspection shows that we can write $\mathscr{L}_H[u]$ in the alternative and more suggestive form

$$v = \mathscr{L}_H[u] = \mathscr{A}(r,\theta;\mathbf{A})\theta + rH(r,r^2)\mathbf{A}^2\theta + \frac{1}{r^n}\frac{d}{dr}\left[r^{n+1}H(r,r^2)\mathbf{A}\right]\theta, \tag{4.3}$$

where $\mathscr{A}$ denotes the scalar-valued function

$$\begin{aligned}\mathscr{A}(r,\theta;\mathbf{A}) =&[H_r(r,r^2) + 2rH_s(r,r^2)](1 + r^2|\mathbf{A}\theta|^2) + rH(r,r^2)[(n+1)|\mathbf{A}\theta|^2\\ &+ r\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle] - rH_s(r,r^2)(n + r^2|\mathbf{A}\theta|^2).\end{aligned} \tag{4.4}$$

As a useful but side remark note that upon introducing the skew-symmetric matrix field $\mathbf{B} = \dot{\mathbf{Q}}\mathbf{Q}^t$ we can write $\mathscr{L}_H[u] = \mathbf{Q}^t w(x)\mathbf{Q}$ where w now refers to the term on the right of (4.3) with $\mathbf{B}$ replacing $\mathbf{A}$ throughout. Now proceeding with (4.2)-(4.3) and noting the PDE $\mathscr{L}_H[u] = \nabla\mathscr{P}$ it follows that

$$\int_0^{2\pi} \langle \mathscr{L}_H[u](r\gamma(t)), \gamma'(t)\rangle\, dt = \int_0^{2\pi} \langle v(r\gamma(t)), \gamma'(t)\rangle\, dt = \int_0^{2\pi} \frac{d}{dt}\mathscr{P}(\gamma(t))\, dt = 0 \tag{4.5}$$

(with prime denoting d/dt) where $\gamma = \gamma(t) \in \mathscr{C}^1([0,2\pi],\mathbb{S}^{n-1})$ is *closed* and $x = r\gamma(t)$ with $a < r < b$ fixed. Henceforth we assume this to be true and look to recover information on the matrix field $\mathbf{A}$. Indeed specialising to $\theta = \gamma(t)$ as above and using (4.3) we can expand the integrand in the left-hand side of (4.5) as

$$\begin{aligned}\langle \mathscr{L}_H[u](r\gamma(t)), \gamma'(t)\rangle =& \langle v(r\gamma(t)), \gamma'(t)\rangle = \mathscr{A}(r,\theta;\mathbf{A})\langle \gamma(t), \gamma'(t)\rangle \\ &+ rH(r,r^2)\langle \mathbf{A}^2\gamma(t), \gamma'(t)\rangle + \frac{1}{r^n}\langle d/dr\left[r^{n+1}H(r,r^2)\mathbf{A}\right]\gamma(t), \gamma'(t)\rangle.\end{aligned} \tag{4.6}$$

Since γ is a curve on the unit sphere we have $\langle \gamma, \gamma'\rangle = 0$ and subsequently (4.5)-(4.6) under the assumption $v = \mathscr{L}_H[u] = \nabla\mathscr{P}$ simplifies to

$$\begin{aligned}&\int_0^{2\pi} \langle \mathscr{L}_H[u](r\gamma(t)), \gamma'(t)\rangle\,dt \\ &= \int_0^{2\pi} \langle \mathbf{E}(r)\gamma(t), \gamma'(t)\rangle\,dt + \frac{1}{r^n}\int_0^{2\pi} \langle \mathbf{F}(r)\gamma(t), \gamma'(t)\rangle\,dt = \int_0^{2\pi} \frac{d}{dt}\mathscr{P}(\gamma(t))\,dt = 0,\end{aligned} \tag{4.7}$$

where $\mathbf{E} = rH(r,r^2)\mathbf{A}^2$ and $\mathbf{F} = d/dr[r^{n+1}H(r,r^2)\mathbf{A}]$ are symmetric and skew-symmetric matrix fields on $]a,b[$ respectively.

LEMMA 2. *Let* $\mathbf{E}$ *be a symmetric* $n \times n$ *matrix and* $\gamma = \gamma(t) \in \mathscr{C}^1([0,2\pi],\mathbb{S}^{n-1})$ *be a closed curve. Then* $\langle \mathbf{E}\gamma(t), \gamma'(t)\rangle$ *integrates to zero on* $[0,2\pi]$.

Proof. As γ is closed and $\mathbf{E}$ is symmetric this follows by integrating the identity $d/dt\langle \mathbf{E}\gamma, \gamma\rangle = \langle \mathbf{E}\gamma', \gamma\rangle + \langle \mathbf{E}\gamma, \gamma'\rangle = 2\langle \mathbf{E}\gamma, \gamma'\rangle$ noting $\langle \mathbf{E}\gamma, \gamma\rangle|_{t=2\pi} = \langle \mathbf{E}\gamma, \gamma\rangle|_{t=0}$. □

Moving on now, upon utilising this lemma, the integral involving the symmetric matrix field $\mathbf{E} = rH(r,r^2)\mathbf{A}^2$ in (4.7) is seen to vanish and so, summarising, if $v = \mathscr{L}_H[u] = \nabla\mathscr{P}$, then

$$\int_0^{2\pi} \langle \mathscr{L}_H[u](r\gamma(t)), \gamma'(t)\rangle\,dt = \frac{1}{r^n}\int_0^{2\pi} \langle d/dr\left[r^{n+1}H(r,r^2)\mathbf{A}\right]\gamma(t), \gamma'(t)\rangle\,dt = 0 \tag{4.8}$$

for every closed curve $\gamma \in \mathscr{C}^1([0,2\pi],\mathbb{S}^{n-1})$. Now we turn into dealing with the skew-symmetric matrix field $\mathbf{F}$.

LEMMA 3. *Let* $\mathbf{F}$ *be an* $n \times n$ *skew-symmetric matrix and let* $\gamma = \mathbf{PR}\rho$ *with* $\mathbf{P}, \mathbf{R} \in \mathbf{O}(n)$ *and* $\rho \in \mathscr{C}^\infty([0,2\pi],\mathbb{S}^{n-1})$ *the closed curved given by*

$$\rho(t) = \begin{cases} \rho_1 = \sin t\,\sin\phi_2\,\sin\phi_3\ldots\sin\phi_{n-1}, \\ \rho_2 = \cos t\,\sin\phi_2\,\sin\phi_3\ldots\sin\phi_{n-1}, \\ \rho_3 = \cos\phi_2\,\sin\phi_3\ldots\sin\phi_{n-1}, \\ \vdots \\ \rho_{n-1} = \cos\phi_{n-2}\,\sin\phi_{n-1}, \\ \rho_n = \cos\phi_{n-1}. \end{cases} \tag{4.9}$$

Here $\phi_\ell \in [0,\pi]$ for all $2 \leqslant \ell \leqslant n-1$ and denoting by $(e_k : 1 \leqslant k \leqslant n)$ the standard basis of $\mathbb{R}^n$, $\mathbf{R} = \mathbf{R}(i,j)$ is the orthogonal transformation swapping the pair of basis vectors (e_1, e_2) with (e_i, e_j) $(1 \leqslant i < j \leqslant n)$ and leaving the rest fixed. Then $\langle \mathbf{F}\gamma(t), \gamma'(t)\rangle$ integrates to zero on $[a,b]$ $\iff$ $\mathbf{F} = 0$.

Proof. As $\mathbf{F}$ is skew-symmetric we can write $\mathbf{F} = \mathbf{P}\mathbf{D}\mathbf{P}^t$ where $\mathbf{P} \in \mathbf{O}(n)$ and $\mathbf{D} = diag(d_1\mathbf{J}, \dots, d_k\mathbf{J})$ if $n = 2k$ and $\mathbf{D} = diag(d_1\mathbf{J}, \dots, d_{k-1}\mathbf{J}, 0)$ if $n = 2k-1$. Here $\mathbf{J}$ is the 2×2 rotation matrix by angle $\pi/2$ [*cf.* (2.3)]. Now setting $\gamma = \mathbf{P}\mathbf{R}\rho$ we have

$$\int_0^{2\pi} \langle \mathbf{F}\gamma(t), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \langle \mathbf{D}\omega(t), \omega'(t)\rangle dt = 0 \tag{4.10}$$

where $\omega := \mathbf{R}\rho$. Thus to prove the lemma it is sufficient to show that the last equality in (4.10) implies $\mathbf{D} = 0$. Arguing component-wise and substituting ω into (4.10) with $\omega'(t) = \mathbf{R}\rho'(t) = \mathbf{R}(\rho_2, -\rho_1, 0, \dots, 0)$ it is seen that

$$\int_0^{2\pi} \langle \mathbf{D}\omega, \omega'\rangle \, dt = 2\pi(\rho_1^2 + \rho_2^2)\mathbf{D}_{ij}. \tag{4.11}$$

As such if the integral on the left is zero then $\mathbf{D} = 0$. This finishes the proof. □

THEOREM 4. *If a twist u with $\mathbf{Q} \in \mathscr{C}([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$ satisfies $\mathscr{L}_H[u] = \nabla \mathscr{P}$ for some $\mathscr{P}$, then the twist path satisfies the ODE*

$$\frac{d}{dr}\left\{ r^{n+1} H(r, r^2) \mathbf{Q}^t \frac{d\mathbf{Q}}{dr} \right\} = 0, \qquad a < r < b. \tag{4.12}$$

Proof. This follows directly by combining the conclusions of the above two lemmas. □

Let us now look at this last ODE from a different angle. Indeed the energy of an admissible twist u, upon noting $d\mu(x,u) = H(|x|, |x|^2) d\mathscr{L}^n$ can be written,

$$\begin{aligned} \mathbb{I}[u; \mathbb{X}_n] &= \int_{\mathbb{X}_n} |\nabla u|^2 \, d\mu(x,u) = \int_{\mathbb{S}^{n-1}} \int_a^b H(r, r^2)(n + r^2 |\dot{\mathbf{Q}}\theta|^2) r^{n-1} \, dr d\mathscr{H}^{n-1} \\ &= \int_{\mathbb{X}_n} |\nabla x|^2 \, d\mu(x,x) + \omega_n \int_a^b H(r, r^2) |\dot{\mathbf{Q}}|^2 r^{n+1} \, dr. \end{aligned} \tag{4.13}$$

Proceeding forward we now set for each admissible twist loop $\mathbf{Q}$ with associated twist u the energy integral

$$\mathbb{E}[\mathbf{Q}; (a,b)] = \frac{\mathbb{I}[u; \mathbb{X}_n] - \mathbb{I}[x; \mathbb{X}_n]}{\omega_n} = \int_a^b H(r, r^2) |\dot{\mathbf{Q}}|^2 r^{n+1} \, dr. \tag{4.14}$$

The Euler-Lagrange equation for this energy integral over the space of admissible loops $\mathscr{J}(a,b) = \{\mathbf{Q} \in \mathscr{W}^{1,2}(a,b; \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\}$ can then be seen to be the second order ODE for $\mathbf{Q} = \mathbf{Q}(r)$ and equivalent to (4.12):

$$\frac{d}{dr}\left[r^{n+1} H(r, r^2) \frac{d\mathbf{Q}}{dr} \mathbf{Q}^t \right] = \mathbf{Q} \frac{d}{dr}\left[r^{n+1} H(r, r^2) \mathbf{Q}^t \frac{d\mathbf{Q}}{dr} \right] \mathbf{Q}^t = 0. \tag{4.15}$$

The ODE (4.12) and geodesics on $\mathbf{SO}(n)$. We now turn into resolving the boundary value problem associated with the ODE (4.12) over the space of loops $\mathscr{J}(a,b)$ as defined above. Indeed a first integration yields $r^{n+1}H(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}} = \mathbf{H}$ for a constant skew-symmetric matrix $\mathbf{H}$. When combined with the left boundary condition $\mathbf{Q}(a) = \mathbf{I}_n$ this first order ODE is seen to have the general solution $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ where the *profile* $\mathscr{H} \in \mathscr{C}^2[a,b]$ is given by (1.14).

Anticipating on the right boundary condition $\mathbf{Q}(b) = \mathbf{I}_n$, we can proceed by first orthogonally diagonalising $\mathbf{H}$ hence writing $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$ and $\mathbf{H} = \mathbf{P}diag(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},0)\mathbf{P}^t$ when $n = 2k-1$ where $\mathbf{P} \in \mathbf{O}(n)$, and the 2×2 matrices $\mathbf{J}$ and $\mathbf{R}$ are given by (2.3). Verifying the boundary condition $\mathbf{Q}(b) = \mathbf{I}_n$ in even and odd dimensions we then have:

- $(n = 2k)$ Here we write

$$\begin{aligned}\mathbf{Q}(b) &= \exp\{\mathscr{H}(b)\mathbf{H}\} = \exp\{\mathbf{P}diag(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t\}\\ &= \mathbf{P}diag(\mathbf{R}[c_1],\dots,\mathbf{R}[c_k])\mathbf{P}^t\\ &= \mathbf{I}_n \iff c_j = 2m_j\pi, \qquad m_j \in \mathbb{Z}, \qquad \forall 1 \leqslant j \leqslant k.\end{aligned}$$

- $(n = 2k-1)$ Here we write

$$\begin{aligned}\mathbf{Q}(b) &= \exp\{\mathscr{H}(b)\mathbf{H}\} = \exp\{\mathbf{P}diag(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},0)\mathbf{P}^t\}\\ &= \mathbf{P}diag(\mathbf{R}[c_1],\dots,\mathbf{R}[c_{k-1}],1)\mathbf{P}^t\\ &= \mathbf{I}_n \iff c_j = 2m_j\pi, \qquad m_j \in \mathbb{Z}, \qquad \forall 1 \leqslant j \leqslant k-1.\end{aligned}$$

In conclusion the solutions $\mathbf{Q} = \mathbf{Q}(r;\mathsf{m})$ to (4.12) in $\mathscr{J}(a,b)$ are given by $\mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}\}$, where $\mathscr{H}(r)$ is given by (1.14), $\mathsf{m} = (m_1,\dots,m_k)$ and for the skew-symmetric matrix $\mathbf{H}$ we have

$$\mathbf{H} = \begin{cases}\mathbf{P}diag(2m_1\pi\mathbf{J},\dots,2m_k\pi\mathbf{J})\mathbf{P}^t, & n = 2k,\\ \mathbf{P}diag(2m_1\pi\mathbf{J},\dots,2m_{k-1}\pi\mathbf{J},0)\mathbf{P}^t, & n = 2k-1.\end{cases} \tag{4.16}$$

Here we remark that the resulting twist loops $\mathbf{Q} = \mathbf{Q}(r;\mathsf{m}) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ are *closed* rescaled geodesics on the compact Lie group $\mathbf{SO}(n)$ based at $\mathbf{I}_n$ with the skew-symmetric matrix $\mathbf{H}$ in the Lie algebra $\mathfrak{so}(n)$ presenting the tangent direction at the end-points and the matrix exponential being the canonical exponential map from the Lie algebra $\mathfrak{so}(n)$ to the Lie group $\mathbf{SO}(n)$.

5. Twists as solutions to the system (1.2) and the proof of Theorem 2

In this last section we turn again to the differential operator $\mathscr{L}_H$ and seek solutions to the nonlinear system (1.2) in the form of twists. As here necessarily the twist loop $\mathbf{Q}$ solves the ODE $d/dr[r^{n+1}H(r,r^2)\mathbf{Q}^t\dot{\mathbf{Q}}] = 0$ in $\mathscr{J}(a,b)$, in view of what was discussed in the previous section, we must have $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ for $a \leqslant r \leqslant b$ with $\mathscr{H} =$

$\mathscr{H}(r)$ as in (1.14) and $\mathbf{H}$ as in (4.16). Now the action of $\mathscr{L}_H$ on u given by (4.1) or (4.3)-(4.4) simplifies to

$$\begin{aligned}\mathscr{L}_H[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] &= \nabla H(|x|,|x|^2) - nrH_s(r,r^2)\theta \\ &+ \left[r^2H_r(r,r^2) + r^3H_s(r,r^2) + r(n+1)H(r,r^2)\right]\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\ &+ r^2H(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2\theta + rH(r,r^2)\dot{\mathscr{H}}^2\mathbf{H}^2\theta. \end{aligned} \tag{5.1}$$

With this introduction and formulation at hand we can now completely describe the twist solutions to the system (1.2)-(1.3) and present the proof Theorem 2.

Proof of Theorem 2. : Referring to (5.1) it is seen by inspection that the first two terms on the right are gradient fields and so we can focus on the remainder

$$\begin{aligned}U(x) =& \mathscr{L}_H[u] - \nabla H(|x|,|x|^2) + nH_s(|x|,|x|^2)x \\ =& \mathscr{A}(|x|,|\mathbf{H}x|^2)x + \mathscr{B}(|x|,|\mathbf{H}x|^2)\mathbf{H}^2x,\end{aligned} \tag{5.2}$$

where $\mathscr{A}(r,z) = -H_s(r,r^2)\dot{\mathscr{H}}^2z$ and $\mathscr{B}(r,z) = H(r,r^2)\dot{\mathscr{H}}^2$. Here we have used the fact that $\mathscr{H}$ satisfies the ODE $d/dr[r^{n+1}H(r,r^2)\dot{\mathscr{H}}(r)] = 0$ to simplify the expression for $\mathscr{A}$. Referring to Lemma 1 an initial inspection now shows that

$$\begin{aligned}2\mathscr{A}_z(r,z) + \frac{\mathscr{B}_r(r,z)}{r} &= \frac{1}{r}H_r(r,r^2)\dot{\mathscr{H}}^2 + \frac{2}{r}H(r,r^2)\dot{\mathscr{H}}\ddot{\mathscr{H}} \\ &= -\frac{\dot{\mathscr{H}}^2}{r^2}\left[2(n+1)H(r,r^2) + rH_r(r,r^2) + 4r^2H_s(r,r^2)\right]\end{aligned} \tag{5.3}$$

where in concluding the second line we have made further use of the above ODE to substitute for $\ddot{\mathscr{H}}$. As before let $\mathscr{F}_H(r) = rH_r(r,r^2) + 2(n+1)H(r,r^2) + 4r^2H_s(r,r^2)$. Then if $\mathscr{F}_H \not\equiv 0$ on $]a,b[$ by invoking the first part of Lemma 1 with $\mathscr{A}_z + \mathscr{B}_r/r \not\equiv 0$ we have:

$$\operatorname{curl}\mathscr{L}_H[u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta] = 0 \iff \mathbf{H}^2 = -c^2\mathbf{I}_n \iff \mathscr{L}_H[u] \text{ is a gradient.}$$

This, given the orthogonal diagonalisation of the skew-symmetric matrix $\mathbf{H}$ from above and (4.16) lead to $\mathbf{Q} \equiv \mathbf{I}_n$ when n is odd and $\mathbf{Q}(r) = \exp\{2m\pi\mathscr{H}(r)\mathbf{P}\mathbf{J}_n\mathbf{P}^t\}$ when n is even, where $m \in \mathbb{Z}$ and $\mathbf{J}_n = diag(\mathbf{J},\dots,\mathbf{J})$ with $\mathbf{J}$ as in (2.3).

Next when $\mathscr{F}_H \equiv 0$ on $]a,b[$ the corresponding vector field $\mathscr{L}_H[u]$ is still a gradient by the second part of Lemma 1 but now with no further restrictions on the skew-symmetric matrix $\mathbf{H}$. Indeed more directly to show that U is a gradient and hence $u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta$ solves $\mathscr{L}_H[u] = \nabla\mathscr{P}$ we show that there exists $f = f(r,z) \in \mathscr{C}^2([a,b]\times\mathbb{R},\mathbb{R})$ such that

$$\nabla f(|x|,|\mathbf{H}x|^2) = f_r(|x|,|\mathbf{H}x|^2)x/|x| - 2f_z(|x|,|\mathbf{H}x|^2)\mathbf{H}^2x = U(x), \qquad x \in \mathbb{X}_n.$$

By direct verification it is seen that (up to an additive constant) the function $f(r,z) = -1/2H(r,r^2)\dot{\mathscr{H}}^2z$ with $a \leqslant r \leqslant b$ and $z \in \mathbb{R}$ satisfies these conditions. Thus $U = \nabla f(|x|,|\mathbf{H}x|^2)$ and so $\mathscr{L}_H[u]$ is a gradient field. $\square$

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